A Procedure for Evaluating the Spacewise Variations of Continuous Turbulence on Airplane Responses

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As airplanes become larger and more flexible, they become more susceptible to dynamic excitation due to spacewise variations in gust loading over the major aerodynamic surfaces. An analytical procedure is derived to account for spacewise variations in gust velocity on airplane dynamic responses. The gust velocity field is assumed to be random in the stationary Gaussian sense, and is described in three orthogonal components by appropriate cross power spectral density functions. The cross power spectral density functions are derived from a standard form of gust velocity autocorrelation function. This assumption implies that the turbulence model is frozen in time, random in space. The autocorrelation function is used to generate a gust spectral matrix for use in airplane dynamic gust analysis. Finally, the procedure for obtaining the gust spectral matrix, which is a matrix of gust cross power spectral density functions, is reduced to use of a single chart and simple arithmetic. This procedure is ideally suited for digital computer "look-up" and computation. The gust spectral matrix is wholly consistent with the one-dimensional power spectral analysis procedures in current use, and will reduce to identical results when variations in gust velocity normal to direction of flight are ignored. The application of this procedure to airplane continuous turbulence analyses requires the computation of the complex frequency response functions for gust loading on individual aerodynamic panels.

Introduction

IF airplane flutter model tests or analysis show low damping in an asymmetrical mode at cruise velocities, or if the wing tended to respond significantly to lateral gust excitation, a question is raised concerning the spanwise variation and interaction effects of the lateral and vertical components of continuous turbulence. It is not the purpose herein to develop the aeroelastic dynamic equations of motion, the accompanying airplane load equations, or to defend or support the use of power spectral gust analysis techniques to arrive at airplane design loads. Many excellent texts and papers have already been written on these subjects.[1,2]

Rather, the purpose is to indicate how airplane power spectral gust analysis techniques currently being used in industry, government agencies, and at various universities can be extended to account for the spacewise variations of continuous turbulence with only moderate additional effort. It will be noted that separate lateral and vertical analysis results can be combined in a single analysis if appropriate degrees of freedom are included. Consequently, the necessary equation will be indicated only as having matrix form. It is assumed that the equations of motion include sufficient degrees of freedom such as rigid body freedoms (vertical translation, lateral translation, pitch, yaw, roll), and elastic freedoms (symmetrical wing modes, antisymmetrical wing modes, fuselage vertical bending, fuselage side bending, fuselage torsion, fin bending, fin torsion).

Equations of Motion

The equations of motion can be written in one or more components of turbulence u, v, w in the fore and aft, lateral or vertical directions, respectively, as indicated in Eq. (1);

$$[M_3]\{\ddot{q}\} + [M_2]\{\dot{q}\} + [M_1]\{q\} = [C]\begin{Bmatrix} u \\ \cdot \\ \cdot \\ \cdot \\ v \\ \cdot \\ \cdot \\ \cdot \\ w \\ \cdot \\ \cdot \end{Bmatrix} \tag{1}$$

Note that the column matrix on the right-hand side of Eq. (1) gives the gust velocity time histories for all of the n aerodynamic panels on the airplane. The matrix C is a generalized gust airforce matrix relating the generalized force due to gust excitation on each aerodynamic panel to the generalized coordinate responses. The corresponding load equations are given in Eq. (2);

$$\begin{Bmatrix} \cdot \\ \cdot \\ l \\ \cdot \end{Bmatrix} = [\bar{M}_3]\{\ddot{q}\} + [\bar{M}_2]\{\dot{q}\} + [\bar{M}_1]\{q\} + [\bar{C}]\begin{Bmatrix} u \\ \cdot \\ \cdot \\ \cdot \\ v \\ \cdot \\ \cdot \\ \cdot \\ w \\ \cdot \\ \cdot \end{Bmatrix} \tag{2}$$

The load equations utilize the same gust column matrix and a k by n matrix $\bar{C}$ relating the gust forces on each of the n panels to the various load, stress, deflection, or other output quantity of interest in the column matrix l on the left. The

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matrices in the equations of motion are square matrices wherein the number of rows and columns are equal to the number of degrees of freedom j. The $\bar{M}$ matrices in the load equations are rectangular matrices with k rows and j columns, wherein k is the number of load quantities of interest. Both the equations of motion and the load equations are solvable in this form for specified time histories of gust velocity on all aerodynamic panels.

Response to Random Loading

Consider that we make all elements of the column gust matrix in Eqs. (1) and (2) equal to zero except one element, and that this element gust time history is replaced by a unit impulse $\delta(t)$. Then, it is possible to solve for all generalized coordinate impulse responses $q_\delta(t)$ and all of the load impulse responses $l_\delta(t)$. If the impulse responses were determined for a unit impulse on each of the n aerodynamic panels in turn, we could obtain generalized coordinate and load impulse response column matrices.

If we consider unit gust velocities on each aerodynamic panel and take the Laplace transform of the resulting equations, we can obtain the Laplace transforms of the generalized coordinate impulse response column matrix as shown in Eq. (3) and the load impulse response matrix in Eq. (4);

$$[(s^2M_3) + (sM_2) + (M_1)]\{\mathfrak{L}(q_\delta)\} = [C]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

$$[B(s)]\{\mathfrak{L}(q_\delta)\} = [C]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix} \tag{3}$$

$$\{\mathfrak{L}(q_\delta)\} = [B(s)]^{-1}[C]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

$$\{\mathfrak{L}(l_\delta)\} = [(s^2\bar{M}_3) + (s\bar{M}_2) + (\bar{M}_1)]\{\mathfrak{L}(q_\delta)\} + [\bar{C}]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix} \tag{4}$$

$$\{\mathfrak{L}(l_\delta)\} = \{[\bar{B}(s)][B(s)]^{-1}[C] + [\bar{C}]\}\begin{Bmatrix} \delta \\ \cdot \\ \cdot \\ \delta \end{Bmatrix} = [P(s)]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

A matrix of load impulse response time histories is obtained for the inverse transform of Eq. (4);

$$\{l_\delta(t)\} = \mathfrak{L}^{-1}\{\mathfrak{L}(l_\delta)\} = \mathfrak{L}^{-1}[P(s)]\begin{Bmatrix} 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix} = [P(t)]\begin{Bmatrix} \delta \\ \cdot \\ \cdot \\ \delta \end{Bmatrix} \tag{5}$$

A matrix $l(t)$ of load responses to prescribed time histories of gust velocity at all panels can be expressed in terms of the Duhamel integral

$$\{l(t)\} = \int_0^t [P(t-\tau)]\begin{Bmatrix} u(\tau) \\ \cdot \\ \cdot \\ v(\tau) \\ \cdot \\ \cdot \\ w(\tau) \end{Bmatrix} d\tau \tag{6}$$

If we take the direct and conjugate Fourier transforms of Eq. (6), we obtain

$$\mathfrak{F}\{l(t)\} = \int_{-\infty}^{\infty} \{l(t)\}e^{i\omega t}dt$$

$$= \int_{-\infty}^{\infty} [P(t-\tau)]e^{i\omega(t-\tau)}dt \int_0^t \begin{Bmatrix} u(\tau) \\ \cdot \\ \cdot \\ v(\tau) \\ \cdot \\ \cdot \\ w(\tau) \end{Bmatrix} e^{i\omega\tau}d\tau \tag{7a}$$

$$\mathfrak{F}^*\{l(t)\} = \int_{-\infty}^{\infty} \{l(t)\}e^{i\omega t}dt$$

$$= \int_{-\infty}^{\infty} [P(t-\tau)]e^{-i\omega(t-\tau)}dt \int_0^t \begin{Bmatrix} u(\tau) \\ v(\tau) \\ w(\tau) \end{Bmatrix} e^{-i\omega\tau}d\tau \tag{7b}$$

The spectral matrix for the loads is determined by extending the limits of integration on the integrals with respect to τ in Eqs. (7a) and (7b) to $+\infty$ and $-\infty$ and taking the product of the two equations;

$$[\phi_l(i\omega)] = \lim_{|\tau|\to\infty} \mathfrak{F}\{l(t)\}^*\mathfrak{F}\lfloor l(t)\rfloor \tag{8}$$

$$[\phi_l(i\omega)] = [P(i\omega)]\begin{Bmatrix} u(i\omega) \\ \cdot \\ \cdot \\ v(i\omega) \\ \cdot \\ \cdot \\ w(i\omega) \end{Bmatrix}\lfloor u^*(i\omega)\cdot v^*(i\omega)\cdot w^*(i\omega)\rfloor[P^*(i\omega)]$$

$$[\phi_l(i\omega)] = [P(i\omega)][\phi_g(i\omega)][P^*(i\omega)]$$

It will be noted that $P^*(i\omega)$ is the conjugate transpose of $P(i\omega)$. The matrix $P(i\omega)$ is a matrix of complex frequency response functions relating each load quantity to sinusoidal gust excitation on each of the separate aerodynamic panels.

The matrix $\phi_g(i\omega)$ is the matrix of gust cross spectral density functions relating the gust time histories for all pairs of aerodynamic panels. Of course, each diagonal element of this matrix is the gust power spectral density function.

The load spectral matrix is a matrix of cross spectral density functions relating the load time histories for all pairs of load quantities. Not all of the off-diagonal elements of this matrix, the load cross spectra, are of interest. The off-diagonal elements or load cross spectra relating load components at a given station on the airplane may be of interest of combined loads are being considered, such as the interaction of shear, moment, and torsion on the wing.[3] The diagonal elements, of course, are the power spectral density functions for the individual load quantities and are of primary interest.

The load spectral matrix will include the effects of all components of the gust loading that were included in the gust excitation matrix in Eqs. (1) and (2). That is, it can account for gusts acting on the airplane in any direction and with various distributions over the airplane;

Gust Spectral Matrix

The gust spectral matrix will have the form indicated in Eq. (9) if the turbulence is assumed to be isotropic;

$$[\phi_g(i\omega)] = \left[\begin{array}{cc:cc:cc} \phi_{u_{11}} & \phi_{u_{12}} & 0 & & 0 & \\ \phi_{u_{21}} & \phi_{u_{22}} & & & & \\ \hdashline & 0 & \phi_{v_{33}} & \phi_{v_{34}} & 0 & \\ & & \phi_{v_{43}} & \phi_{v_{44}} & & \\ \hdashline & 0 & 0 & & \phi_{w_{55}} & \phi_{w_{56}} \\ & & & & \phi_{w_{65}} & \phi_{w_{66}} \end{array}\right] \tag{9}$$

If the turbulence is assumed to be isotropic,[4] then

$$\phi_{uv} = \phi_{uw} = \phi_{vw} = 0$$
$$\phi_{vu} = \phi_{wu} = \phi_{wv} = 0 \tag{10}$$

and the equations of motion and load equations are not coupled by the gust excitation between fore and aft, lateral, and vertical excitation. If the responses and loads are also uncoupled, the matrix equations can be partitioned and solved separately for each turbulence component. Then, the resulting root-mean-square load values $\bar{A}$, and the zero crossings N_0 can be combined later to arrive at a final result.

The gust spectral matrix for "frozen" isotropic turbulence can be generated for any system of aerodynamic panels after observing that the space correlation is dependent only upon the space separation. If we consider two aerodynamic panels having coordinates relative to the airplane reference axes (x_1, y_1, z_1) and (x_2, y_2, z_2), then the correlation distance r' can be determined;

$$r' = [(x_1 - x_2 + V\tau)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} \quad (11)$$

In single dimensional gust load studies we have observed that the autocorrelation function $R(\tau)$ is a function of τ, where

$$r = V\tau \quad (12)$$

Of course, the Fourier transform of the autocorrelation function is the power spectrum $\phi(\omega)$;

$$\phi(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(\tau)e^{i\omega\tau}d\tau \quad (13)$$

The space-time correlation function for the u, v, or w component of turbulence for two panels, say panels 1 and 2, can be determined by the transformation of Eq. (13) by Eqs. (11) and (12);

$$\phi_{12}\left(\frac{\omega}{V}\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_{12}(r')e^{(i\omega r/V)}\frac{dr}{V} \quad (14)$$

If the reduced frequency Ω is used where $\Omega = \omega/V$, then Eq. (14) becomes

$$\phi_{12}(i\Omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_{12}(r')e^{i\Omega r}\,dr$$

$$\phi_{12}(i\Omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_{12}(r')\cos\Omega r dr + \frac{i}{\pi}\int_{-\infty}^{\infty} R_{12}(r')\sin\Omega r dr \quad (15)$$

Simple numerical integration techniques can be used to compute the off-diagonal elements of the gust spectral matrix.

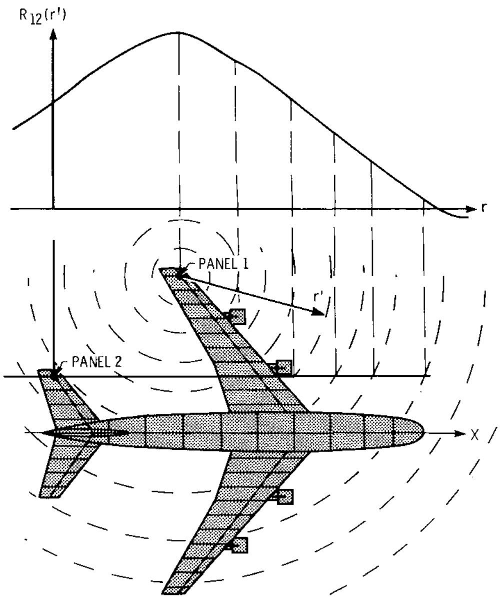

Fig. 1 Procedure for determining space-time correlation function.

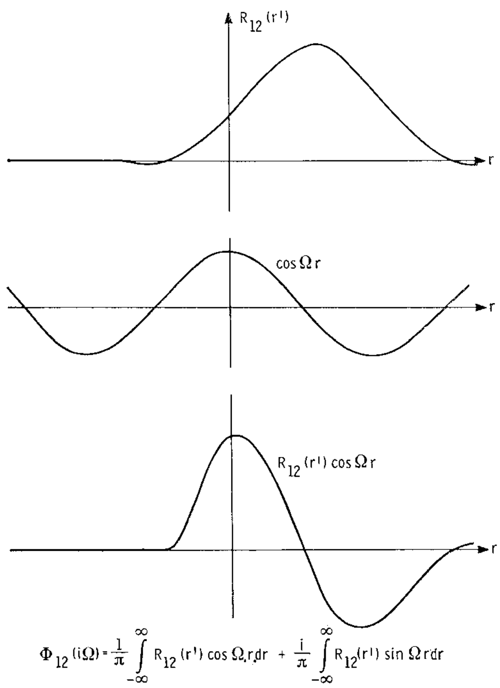

Fig. 2 Evaluation of the cross spectral density function.

Since it is known that the gust matrix is symmetric for the real parts (cospectra) and skew symmetric for the imaginary parts (quadrature spectra), it is only necessary to compute the off-diagonal elements on one side of the diagonal.

The procedure for determining the space-time correlation function $R_{12}(r')$ is shown in Fig. 1, where $(z_1 - z_2)$ is assumed to be equal to zero. The space correlation function $R(r')$ is centered on panel 1. A vertical plane parallel to the x axis is passed through $R(r')$ at panel 2. The resulting section is $R_{12}(r')$, and the origin is at panel 2.

The diagrams in Fig. 2 show how the cospectrum is evaluated for a given reduced frequency Ω. The quadrature spectrum is evaluated in a similar manner using the sine in place of the cosine function as indicated.

If the foregoing procedure is used to calculate the matrix of gust cross spectral density functions for a rectangular array of points in space, equal levels of cospectral and quadrature spectral density relative to a single reference point can be obtained for each frequency. That is, normalized spectral influence charts can be constructed. A typical example is shown in Fig. 3 for the von Karman isotropic turbulence power spectrum[5]

$$\phi_g(\Omega) = \frac{\sigma^2 L}{\pi}\,\frac{1 + \frac{8}{3}(1.339\Omega L)^2}{[1 + (1.339\Omega L)^2]^{11/6}} \quad (16)$$

The use of spectral influence charts to construct the gust spectral matrix $\phi_g(i\Omega)$ suggests the use of a normalized gust spectral matrix $\psi_g(i\Omega L)$;

$$[\phi_g(i\Omega L)] = \phi_g(\Omega L)[\psi_g(i\Omega L)] \quad (17)$$

The charts, such as that shown in Fig. 3, are used by drawing the airplane to the proper scale depending on the reduced frequency Ω and the scale of turbulence L, and placing a reference point, say panel 1 indicated in Fig. 1, on the reference point for cospectrum or quadrature spectrum. Then, the normalized cospectral or quadrature spectral densities for the given reduced frequency for all other panels can be read from the chart. When these values are multiplied by the power spectral density, Eq. (16), the cospectral and quadrature spectral densities are obtained.

The period of repetition of the influence chart in the direction of flight, left to right, is $2\pi/\Omega$ or r/L of $2\pi/\Omega L$. Therefore, the dimensioned distance $\pi/2\Omega L$, shown in Fig. 3, repre-

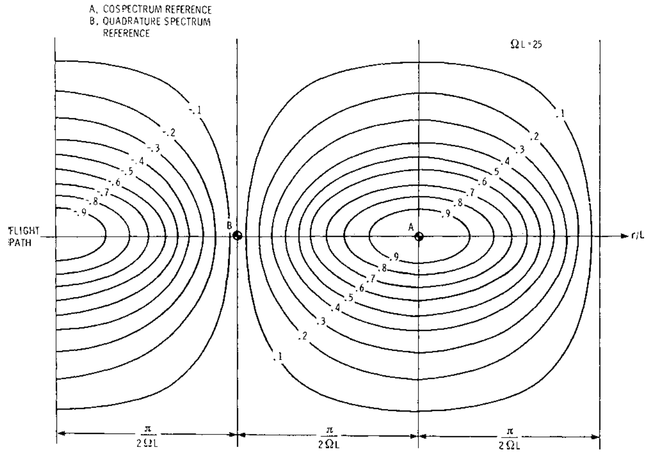

Fig. 3 Typical influence chart for normalized cross spectral densities.

sents $\pi L/2(\Omega L)$ or $\pi L/50$ ft, since the chart is for ΩL of 25. The same scale is used in the y direction.

It will be noted that the assumption of frozen turbulence implies that the influence charts should be repeated endlessly at intervals r/L or $2\pi/\Omega L$, as indicated in Fig. 4. However, if the turbulence is assumed to decay, the influence from adjacent periods probably should be neglected.

An understanding of the influence chart approach provides the engineer with an intuitive feeling of the importance of spacewise variations in turbulence. Obviously, if the airplane is small and the scale of turbulence is large, the spectral densities will vary only slightly from wing tip to wing tip, and spanwise effects can safely be ignored. On the other hand, if the airplane is large and is operating in regimes where the turbulence scale is small, then spanwise effects may be important.

An influence chart approach is somewhat cumbersome because a separate chart is required for each reduced frequency; however, an alternate approach is available. It was noted that the influence chart has a period in the flight or r direction of $2\pi/\Omega$, and it can be shown that for a given y separation between points on the airplane, both the cospectrum and quadrature spectrum vary harmonically for different x separations. Consider an arbitrary x separation between points 1 and 2 of k/Ω where the period is $2\pi/\Omega$. Then, the cross spectral density is

$$\begin{aligned}\phi_{12}(i\Omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} R(r')e^{i\Omega(r+k/\Omega)}\,dr \\ &= \frac{e^{ik}}{\pi}\int_{-\infty}^{\infty} R(r')e^{i\Omega r}\,dr \\ &= \frac{(\cos k + i\sin k)}{\pi}\int_{-\infty}^{\infty} R(r')e^{i\Omega r}\,dr\end{aligned} \tag{18}$$

Therefore, the cospectral density can be computed for various y separations and reduced frequencies with zero x separation, and both the cospectral density and quadrature spectral density can be determined. Where

$$\begin{aligned}C_{12}(\Omega) &= \phi_g(\Omega)\cos\Omega(x_1 - x_2)\psi_g(\Omega) \\ Q_{12}(\Omega) &= \phi_g(\Omega)\sin\Omega(x_1 - x_2)\psi_g(\Omega)\end{aligned} \tag{19}$$

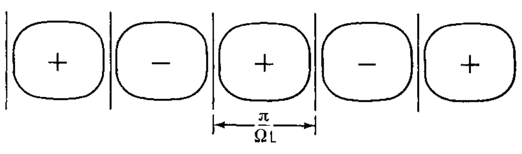

Fig. 4 Influence chart for frozen turbulence.

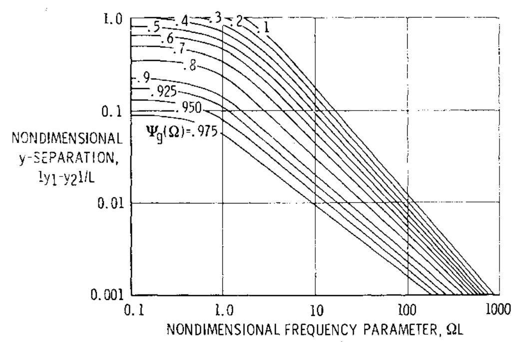

Fig. 5 Nondimensional cospectral density, zero x separation.

$\psi_g(\Omega)$ is the normalized cospectral density for zero x separation and a given y separation.

Values of $\psi_g(\Omega)$ are plotted vs the frequency parameter ΩL, and the y separation $|y_1 - y_2|/L$ in Fig. 5.

Concluding Remarks

A procedure has been outlined to show how current airplane gust analysis techniques can be extended to account for the spacewise variations of continuous turbulence. Only moderate additional effort is required to account for the spacewise variations in one, two, or three components of turbulence.

The spacewise effects of turbulence are embodied in a gust spectral matrix which accounts for the phasing as well as the magnitude of the gust excitation on all pairs of aerodynamic panels. The gust spectral matrix is derived on the assumption that the turbulence is frozen; that is, it remains unchanged in time but varies in space. This assumption results from using a gust autocorrelation function to generate the space correlation function. This approach was also followed by Houbolt.[5] The determination of the gust spectral matrix can be reduced to the use of influence charts, one for each reduced frequency, together with layouts of the airplane.

An alternate method for generating the gust spectral matrix is given by showing that both the cospectral and quadrature spectral densities in the gust spectral matrix vary harmonically in the flight direction with a period of $2\pi/\Omega$. Therefore, both the cospectral and quadrature spectral densities can be obtained from knowledge of the cospectral density function for zero x separation. Finally, a nondimensional plot is given for normalized cospectral density for zero x separation. This type of information can be stored in a computer and employed in automatic digital computation of the gust spectral matrix using standard look-up procedures.

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